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Topology and its Applications 155 (2008) 522–526

Topology and its Applications

www.elsevier.com/locate/topol

On finite unions of certain D-spaces ☆

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Received 30 May 2007; received in revised form 5 November 2007; accepted 18 November 2007

Abstract

In this note, we show that if X is the union of a finite collection $\{X_i\colon i = 1, \dots, k\}$ of strong Σ-spaces, then X is a D-space. As a corollary, we get a conclusion that if X is the union of a finite collection $\{X_i\colon i = 1, \dots, k\}$ of Moore spaces, then X is a D-space. This gives a positive answer to one of Arhangel'skii's problems [A.V. Arhangel'skii, D-spaces and finite unions, Proc. AMS 132 (7) (2004) 2163–2170]. In the last part of the note, we show that if X is the union of a finite collection $\{X_i\colon i = 1, \dots, k\}$ of **DC**-like spaces, then X is a D-space, where **DC** is the class of all discrete unions of compact spaces. As a corollary, we show that if X is the union of a finite collection of regular subparacompact **C**-scattered spaces, then X is a D-space.
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MSC: 54F99

Keywords: D-space; **DC**-like space; Moore space; Strong Σ-space

Introduction

The notion of D-space was introduced by van Douwen (cf. [5]). A *neighborhood assignment* for a space X is a function ϕ from X to the topology of the space X, such that $x \in \phi(x)$ for any $x \in X$. A space X is called a *D-space*, if for any neighborhood assignment ϕ for X there exists a closed discrete subset D of X, such that $X = \bigcup\{\phi(d)\colon d \in D\}$ (cf. [5] and [6]). By results of [3], we know that all semi-stratifiable spaces are D-spaces. So we know that all metrizable spaces, all Moore spaces are D-spaces. In [4], Buzyakova proved that every strong Σ-space is a D-space. In 2002, Arhangel'skii and Buzyakova proved that a space with a point-countable base is a D-space (cf. [2]). Peng proved that X is also a D-space if X has a point-countable weak base (cf. [13]). In 2004 (cf. [1]), Arhangel'skii proved that if X is the union of a finite collection $\{X_i\colon i = 1, \dots, k\}$ of spaces with a point-countable base, then X is a D-space. In 2006, Gruenhage proved that if X is a countably compact space and X is a finite union of D-spaces, then X is a compact space (cf. [9]). Peng proved that if X is a countably compact space and X is the union of a countable family of D-spaces, then X is a compact space (cf. [14]).

In [1], Arhangel'skii raised the following problem: Suppose that a space X is the union of a finite collection $\{X_i\colon i = 1, \dots, k\}$ of Moore spaces. Is X a D-space? In this note, we prove that if X is the union of a finite collection

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doi:10.1016/j.topol.2007.11.002

$\{X_i\colon i = 1, \ldots, k\}$ of strong Σ-spaces, then X is a D-space. By results of [10], we see that every Moore space is a σ-space, hence every Moore space is a strong Σ-space. Thus we give a positive answer to the Arhangel'skii's problem.

The class of **DC**-like spaces is defined using a topological game introduced by Galvin. Galvin begins with any closed-hereditary class $\mathbf{K}$ of spaces, i.e. any class of spaces such that if $X \in \mathbf{K}$, then $2^X \subset \mathbf{K}$, where 2^X denotes the collection of all closed subspaces of the space X. Let $\mathbf{C}$ be the class of all compact Hausdorff spaces and let $\mathbf{DC}$ be the class of all discrete union of compact spaces. The topological games $G(\mathbf{K}, X)$ was studied by R. Telgársky in [16]. The concept of $\mathbf{C}$-scattered spaces was introduced in [15]. It was proved that every regular subparacompact $\mathbf{C}$-scattered space is a $\mathbf{DC}$-like space (cf. [16]). In the last part of this note, we prove that if X is a finite union of $\mathbf{DC}$-like spaces, then X is a D-space. By results of [17] and [12], we have the following corollaries: If X is a finite union of regular subparacompact $\mathbf{C}$-scattered spaces, then X is a D-space; If X is a finite union of θ-refinable locally compact spaces, then X is a D-space.

All the spaces in this note are assumed to be Hausdorff spaces. The set of all natural numbers is denoted by N, and ω is $N \cup \{0\}$. In notation and terminology we will follow [7,10,17].

1. On finite unions of strong Σ-spaces

A space X is a *strong Σ-space* if it have a σ-locally finite closed family $\mathcal{F}$ and a cover $\mathcal{K}$ of X by compact sets, such that whenever $K \subset U$ with U open in X and $K \in \mathcal{K}$, then $K \subset F \subset U$ for some $F \in \mathcal{F}$ (cf. [10]). In [4], it is proved that every strong Σ-space is a D-space. In the following, we will discuss the D-property of the union of a finite family of strong Σ-spaces.

Lemma 1. *Suppose $X = \bigcup\{X_i\colon 1 \leqslant i \leqslant n\}$, and $\mathcal{F}$ is a locally finite family of subsets of X_i for some $i \leqslant n$, and $A = \{x\colon x \in X, \mathcal{F}$ is not locally finite at $x\}$. Then A is a closed subset of X and $A \subset X \setminus X_i$.*

Proof. Since $\mathcal{F}$ is a locally finite family of subsets of X_i, we see that $A \subset X \setminus X_i$. For any $x \in \overline{A}$, let U be an open set of X, such that $x \in U$. Thus $U \cap A \neq \emptyset$. Let $z \in U \cap A$. The family $\mathcal{F}$ is not locally finite at z. Thus $|\{F\colon U \cap F \neq \emptyset, F \in \mathcal{F}\}| \geqslant \omega$. So $x \in A$. Thus A is a closed subset of X and $A \subset X \setminus X_i$. □

Lemma 2. *(See [4].) If X is a strong Σ-space, then X is a D-space.*

Theorem 3. *If X is the union of a finite collection $\{X_i\colon 1 \leqslant i \leqslant n\}$ of strong Σ-spaces, then X is a D-space.*

Proof. We argue by induction. If $k = 1$, then X is a D-space by Lemma 2. Assume now that the statement holds for $k = n$, for some $n \in N$, and let us show that it is also true for $k = n + 1$. For each $1 \leqslant i \leqslant n + 1$, let $\mathcal{K}_i$ and $\mathcal{F}_i$ be the families from the definition of a strong Σ-space, they make the subspace X_i to be a strong Σ-space. We may assume that $\mathcal{F}_i = \bigcup\{\mathcal{F}_{(m,i)}\colon m \in N\}$, satisfying that $\mathcal{F}_{(m,i)} \subset \mathcal{F}_{(m+1,i)}$, and $\mathcal{F}_{(m,i)} = \{F^{\alpha}_{(m,i)}\colon \alpha \in \Lambda_{(m,i)}\}$ which is locally finite in X_i, $1 \leqslant i \leqslant n + 1$, $m \in N$. Let ϕ be a neighborhood assignment of X.

We let P be the lexicographical product $(\omega \setminus \{0\}) \times (n + 1 \setminus \{0\})$. For any $p \in P$, there is some $m \in N$ and $i \in n + 1 \setminus \{0\}$, such that $p = (m, i)$. We denote $p(1) = m$ and $p(2) = i$.

Since $\mathcal{F}_{(1,1)}$ is a locally finite family of X_1. We denote $A_{(1,1)} = \{x\colon x \in X, \mathcal{F}_{(1,1)}$ is not locally finite at $x\}$. By Lemma 1, we know that $A_{(1,1)}$ is a closed subset of X, and $A_{(1,1)} \subset X \setminus X_1$. Thus by induction and the fact that D-property is hereditary with respect to closed subsets, we know that $A_{(1,1)}$ is a D-space. So there is a closed discrete subset $D^*_{(1,1)} \subset A_{(1,1)}$, such that $A_{(1,1)} \subset \bigcup\{\phi(d)\colon d \in D^*_{(1,1)}\} = U^*_{(1,1)}$. Let $\mathcal{F}^*_{(1,1)} = \{F^{\alpha *}_{(1,1)}\colon \alpha \in \Lambda_{(1,1)}\}$, where $F^{\alpha *}_{(1,1)} = F^{\alpha}_{(1,1)} \setminus U^*_{(1,1)}$. So $\mathcal{F}^*_{(1,1)}$ is a locally finite family of X.

For each $\alpha \in \Lambda_{(1,1)}$, if there is some compact subset $K^{\alpha}_{(1,1)} \in \mathcal{K}_1$, such that $K^{\alpha}_{(1,1)} \setminus U^*_{(1,1)} \neq \emptyset$, and there is a finite subset $C^{\alpha}_{(1,1)} \subset K^{\alpha}_{(1,1)} \setminus U^*_{(1,1)}$, such that $K^{\alpha}_{(1,1)} \setminus U^*_{(1,1)} \subset F^{\alpha *}_{(1,1)} \subset \bigcup\{\phi(d)\colon d \in C^{\alpha}_{(1,1)}\}$, then we let $D^{\alpha}_{(1,1)} = C^{\alpha}_{(1,1)}$. If there is no such $K^{\alpha}_{(1,1)}$ and $C^{\alpha}_{(1,1)}$, which satisfy the above conditions, then we let $D^{\alpha}_{(1,1)} = \emptyset$. We let $D_{(1,1)} = (\bigcup\{D^{\alpha}_{(1,1)}\colon \alpha \in \Lambda_{(1,1)}\}) \cup D^*_{(1,1)}$. Thus $D_{(1,1)}$ and $D^*_{(1,1)}$ are all closed discrete subsets of X and $D^*_{(1,1)} \subset D_{(1,1)}$.

Let $p \in P$, we assume that we have constructed closed discrete subsets D^*_q and D_q, such that $D^*_q \subset D_q$, and $\bigcup\{\phi(d)\colon d \in \bigcup\{D_l\colon l \leqslant q\}\} = U_q$ for each $q < p$ and $D_q \cap U_l = \emptyset$ for each $l < q$.

In the following we will construct D^*_p and D_p. We assume $p(1) = m$ and $p(2) = i$, where $m \in N$ and $i \in n+1 \setminus \{0\}$.

$\mathcal{F}_p$ is a locally finite family of $X_{p(2)}$, so is the family $\mathcal{F}'_p = \{F_p^{\alpha\prime}\colon \alpha \in \Lambda_p\}$, where $F_p^{\alpha\prime} = F_p^\alpha \setminus (\bigcup\{U_q\colon q < p\})$. Let $A_p = \{x\colon x \in X, \mathcal{F}'_p$ is not locally finite at $x\}$. Thus A_p is a closed subset of X and $A_p \subset X \setminus X_{p(2)}$, and also $A_p \cap (\bigcup\{U_q\colon q < p\}) = \emptyset$. By induction and the fact that D-property is hereditary with respect to closed subspaces. We see that A_p is a D-space. So there is a closed discrete subspace $D_p^* \subset A_p$, such that $A_p \subset \bigcup\{\phi(d)\colon d \in D_p^*\}$. So $D_p^* \cap (\bigcup\{U_q\colon q < p\}) = \emptyset$. Let $U_p^* = (\bigcup\{U_q\colon q < p\}) \cup (\bigcup\{\phi(d)\colon d \in D_p^*\})$ and $\mathcal{F}_p^* = \{F_p^{\alpha *}\colon \alpha \in \Lambda_p\}$, where $F_p^{\alpha *} = F_p^{\alpha\prime} \setminus U_p^*$, and $\alpha \in \Lambda_p$. Thus $\mathcal{F}_p^*$ is a locally finite family of X. For each $\alpha \in \Lambda_p$, if there is a compact set $K_{p(2)}^\alpha \in \mathcal{K}_{p(2)}$, such that $K_{p(2)}^\alpha \setminus U_p^* \neq \emptyset$, and there is a finite subset $C_{p(2)}^\alpha \subset K_{p(2)}^\alpha \setminus U_p^*$, such that $K_{p(2)}^\alpha \setminus U_p^* \subset F_p^{\alpha *} \subset \bigcup\{\phi(d)\colon d \in C_{p(2)}^\alpha\}$, then we let $D_{p(2)}^\alpha = C_{p(2)}^\alpha$. If there is no such $K_{p(2)}^\alpha$ and $C_{p(2)}^\alpha$, which satisfy the above conditions, then we let $D_{p(2)}^\alpha = \emptyset$.

Let $D_p = (\bigcup\{D_{p(2)}^\alpha\colon \alpha \in \Lambda_p\}) \cup D_p^*$. The set $\bigcup\{D_{p(2)}^\alpha\colon \alpha \in \Lambda_p\}$ is a closed discrete subset of X following from the locally finite property of $\mathcal{F}_p^*$. Thus D_p is a closed discrete subset of X and $D_p \cap (\bigcup\{U_q\colon q < p\}) = \emptyset$. We let $U_p = \bigcup\{\phi(d)\colon d \in \bigcup\{D_q\colon q \leqslant p\}\}$. Thus we have closed discrete subsets D_p^* and D_p, satisfying $D_p^* \subset D_p$ and $D_p \subset X \setminus (\bigcup\{\phi(d)\colon d \in \bigcup\{D_q\colon q < p\}\})$.

So we have proved that for each $p \in P$, there are closed discrete subsets D_p^* and D_p, satisfying $D_p^* \subset D_p$ and $D_p \subset X \setminus (\bigcup\{\phi(d)\colon d \in \bigcup\{D_q\colon q < p\}\})$.

Let $D = \bigcup\{D_p\colon p \in P\}$. If we have proved that $X = \bigcup\{\phi(d)\colon d \in D\}$, then we will see that D is a closed discrete subset of X by the property of D_p. In the following we will prove that $\bigcup\{\phi(d)\colon d \in D_p, p \in P\} = X$. Suppose there is some $x \in X \setminus \bigcup\{\phi(d)\colon d \in D_p, p \in P\}$. Since $X = \bigcup\{X_i\colon 1 \leqslant i \leqslant n+1\}$, there is some $1 \leqslant j \leqslant n+1$, such that $x \in X_j$. Since $X_j = \bigcup \mathcal{K}_j$, there is a compact subset $K \in \mathcal{K}_j$, such that $x \in K \setminus \bigcup\{\phi(d)\colon d \in D_p, p \in P\}$. We denote $U_p = \bigcup\{\phi(d)\colon d \in \bigcup\{D_q\colon q \leqslant p\}\}$. So $K \setminus (\bigcup\{U_p\colon p \in P\}) \neq \emptyset$ and $K \setminus \bigcup\{U_p\colon p \in P\}$ is a compact set of X. Thus there is a finite subset $C_K \subset K \setminus \bigcup\{U_p\colon p \in P\}$, such that $K \subset (\bigcup\{\phi(d)\colon d \in C_K\}) \cup (\bigcup\{U_p\colon p \in P\})$. Thus there is some $l \in P$, such that $K \subset (\bigcup\{\phi(d)\colon d \in C_K\}) \cup (\bigcup\{U_q\colon q \leqslant l\}) = O$. Since $K \in \mathcal{K}_j$, so there is some $m \in N$, and $F \in \mathcal{F}_{(m,j)}$, such that $K \subset F \subset O$. Let $(m,j) = q'$, thus there is some $r \in P$, such that $q' < r$ and $l < r$ and $r(2) = j$. So we see that $F \in \mathcal{F}_r$ and $K \subset F \subset (\bigcup\{\phi(d)\colon d \in C_k\}) \cup (\bigcup\{U_q\colon q < r\})$. Thus $K \subset F \subset (\bigcup\{\phi(d)\colon d \in C_k\}) \cup (\bigcup\{U_q\colon q < r\}) \cup (\bigcup\{\phi(d)\colon d \in D_r^*\})$. $C_K \subset X \setminus (\bigcup\{U_q\colon q \leqslant r\})$ and $D_r^* \subset D_r$, thus $C_K \subset X \setminus (\bigcup\{U_q\colon q < r\}) \cup (\bigcup\{\phi(d)\colon d \in D_r^*\})$. We let $(\bigcup\{U_q\colon q < r\}) \cup (\bigcup\{\phi(d)\colon d \in D_r^*\}) = U_r^*$. So $K \setminus U_r^* \subset F \setminus U_r^* \subset \bigcup\{\phi(d)\colon d \in C_K\}$.

Since $F \in \mathcal{F}_r$, we let $F = F_r^\alpha$ for some $\alpha \in \Lambda_r$. By induction, we know that there is some compact $K_{r(2)}^\alpha \in \mathcal{K}_{r(2)}$ and some finite set $C_{r(2)}^\alpha \subset K_{r(2)}^\alpha \setminus U_r^*$, such that $K_{r(2)}^\alpha \setminus U_r^* \subset F_r^\alpha \setminus U_r^* \subset \bigcup\{\phi(d)\colon d \in C_{r(2)}^\alpha\}$. So $D_{r(2)}^\alpha = C_{r(2)}^\alpha \subset D_r$. Thus $x \in F_r^\alpha \subset (\bigcup\{\phi(d)\colon d \in C_{r(2)}^\alpha\}) \cup U_r^* \subset \bigcup\{\phi(d)\colon d \in D_q, q \leqslant r\}$. This contradicts with $x \notin \bigcup\{\phi(d)\colon d \in D_p, p \in P\}$. Thus we have proved that $X = \bigcup\{\phi(d)\colon d \in D\}$.

For any $x \in X$, there is a minimal p, such that $x \in \bigcup\{\phi(d)\colon d \in D_p\} = O_p$ for some $p \in P$. Thus $O_p \cap D_q = \emptyset$ for any $q > p$. Since $\{q\colon q \in P, q < p\}$ is finite, and D_q is a discrete closed set of X for each $q < p$, there is an open set V_x of X, such that $x \in V_x$, and $|V_x \cap (\bigcup\{D_q\colon q \leqslant p\})| \leqslant 1$. Let $O_x = O_p \cap V_x$. Then $x \in O_x$ and $|O_x \cap D| \leqslant 1$. Thus D is a closed discrete subspace of X. So X is a D-space. □

Lemma 4. *(See [10].) Every Moore space is a σ-space, hence every Moore space is a strong Σ-space.*

Corollary 5. *If X is the union of a finite collection $\{X_i\colon i = 1, \ldots, k\}$ of Moore spaces, then X is a D-space.*

Corollary 5 gives a positive answer to one of Arhangel'skii's problems (Problem 1.6 appeared in [1]).

2. About DC-like spaces

We can see the definitions of $G(\mathbf{K}, X)$ and $\mathbf{K}$-like spaces in [16]. The following definitions appeared in [18].

A sequence $(F_0, E_1, F_1, E_2, F_2, \ldots)$ of closed sets in X is a *play* of $G(\mathbf{K}, X)$ if $F_0 = X$ and for each $n \in \omega$:

(1) E_n is the choice of player one,
(2) F_n is the choice of player two,
(3) $E_n \in \mathbf{K}$,
(4) $E_{n+1} \subset F_n$,

(5) $F_{n+1} \subset F_n$,
(6) $E_n \cap F_n = \emptyset$.

Player one wins this play if $\bigcap\{F_n\colon n \in \omega\} = \emptyset$.

A finite sequence $(F_0, E_1, F_1, \ldots, E_n, F_n)$ of closed sets in X is said to be *admissible* for $G(\mathbf{K}, X)$ if each E_i and F_i satisfy the above conditions (1)–(6) for each $1 \leqslant i \leqslant n$ and $F_0 = X$. A function s is said to be a *strategy* for player one in $G(\mathbf{K}, X)$ if the domain of s consists of all the finite sequence $(F_0, F_1, \ldots, F_n)$ of closed sets in X such that $F_0 = X$ and $(E_1, F_1, \ldots, E_n, F_n)$ is admissible for $G(\mathbf{K}, X)$, where $E_i = s(F_0, \ldots, F_{i-1}) \in 2^X \cap \mathbf{K}$, and contained in F_{i-1} for each $1 \leqslant i \leqslant n$.

A strategy s of player one in $G(\mathbf{K}, X)$ is said to be *winning* if it wins each play $(F_0, E_1, F_1, E_2, F_2, \ldots)$, such that $E_n = s(F_0, E_1, F_1, \ldots, F_{n-1})$ for each $n \in N$, where $F_0 = X$, and $(F_0, F_1, \ldots, F_{n-1}, E_n)$ is admissible for $G(\mathbf{K}, X)$. A space X is called a $\mathbf{K}$*-like* space, if player one has a winning strategy in $G(\mathbf{K}, X)$.

The following definition appeared in [18].

A function s from 2^X into $2^X \cap \mathbf{K}$ is a *stationary winning strategy* for player one in $G(\mathbf{K}, X)$ if and only if it satisfies

(i) $s(F) \subset F$ for each $F \in 2^X$, and
(ii) if a decreasing sequence $\{F_n\colon n \geqslant 1\} \subset 2^X$ satisfies that $s(X) \cap F_1 = \emptyset$ and $s(F_n) \cap F_{n+1} = \emptyset$ for each $n \geqslant 1$, then $\bigcap\{F_n\colon n \geqslant 1\} = \emptyset$.

In [8], Galvin and Telgársky proved that player one has a winning strategy in $G(\mathbf{K}, X)$ if and only if it has a stationary winning strategy. So a space X is a $\mathbf{K}$-like space if and only if player one has a stationary winning strategy in $G(\mathbf{K}, X)$.

Lemma 6. *(See [11].) If X is a* $\mathbf{DC}$*-like space, then X is a D-space.*

We know that $\mathbf{DC}$-like spaces are hereditary with respect to closed subspaces.

Theorem 7. *If X is the union of a finite collection $\{X_i\colon 1 \leqslant i \leqslant k\}$ of* $\mathbf{DC}$*-like spaces, then X is a D-space.*

Proof. We prove it by induction. If $k = 1$, then X is a D-space by Lemma 6. Assume now that the statement holds for $k = n$, for some $n \in N$, and let us show that it is also true for $k = n + 1$.

Let ϕ be any neighborhood assignment of X. For each $1 \leqslant i \leqslant n + 1$, let s_i be a stationary winning strategy of player one in $G(\mathbf{DC}, X_i)$. Let $F_0^i = X_i$, $E_1^i = s_i(X_i) \in \mathbf{DC}$. We assume that $s_i(X_i) = \bigcup \mathcal{F}_1^i$, where $\mathcal{F}_1^i$ is a discrete family of compact sets of X_i. Let $A_1^i = \{x\colon x \in X, \mathcal{F}_1^i$ is not locally finite at $x\}$. By Lemma 1, A_1^i is a closed subset of X, and $A_1^i \subset X \setminus X_i$.

By induction we know that A_1^i is a D-space. Thus there is a closed discrete subset $D_1^{i*} \subset A_1^i$, such that $A_1^i \subset \bigcup\{\phi(d)\colon d \in D_1^{i*}\}$. Thus $\mathcal{F}_1^{i*} = \{F^*\colon F^* = F \setminus \bigcup\{\phi(d)\colon d \in D_1^{i*}\}, F \in \mathcal{F}_1^i\}$ is a locally finite family of X, and F^* is a compact subset of X for each $F^* \in \mathcal{F}_1^{i*}$. Let $C_F \subset F^*$, such that $F^* \subset \bigcup\{\phi(d)\colon d \in C_F\}$. Let $D_1^i = (\bigcup\{C_F\colon F \in \mathcal{F}_1^i\}) \cup D_1^{i*}$. Thus D_1^i is a closed discrete subspace of X, and $s_i(X_i) \subset \bigcup\{\phi(d)\colon d \in D_1^i\}$.

Let $D_1 = \bigcup\{D_1^i\colon 1 \leqslant i \leqslant n + 1\}$. Thus D_1 is a closed discrete subset of X and $s_i(X_i) = E_1^i \subset \bigcup\{\phi(d)\colon d \in D_1\}$. Let $F_1^i = X_i \setminus \bigcup\{\phi(d)\colon d \in D_1\}$, $1 \leqslant i \leqslant n + 1$.

For each $1 \leqslant i \leqslant n + 1$, assume we have an admissible sequence $(F_0^i, E_1^i, F_1^i, \ldots, E_m^i, F_m^i)$, for some $m \in N$, and closed discrete subsets D_j, $j \leqslant m$, satisfying $D_{j+1} \cap (\bigcup\{\phi(d)\colon d \in D_p, p \leqslant j\}) = \emptyset$ for each $j \leqslant m - 1$. $F_j^i = X_i \setminus \bigcup\{\phi(d)\colon d \in \bigcup\{D_p\colon p \leqslant j\}\}$ and $E_j^i \subset \bigcup\{\phi(d)\colon d \in D_j\}$.

In the following we will construct D_{m+1}. For each $1 \leqslant i \leqslant n + 1$, $F_m^i = X_i \setminus \bigcup\{\phi(d)\colon d \in \bigcup\{D_p\colon p \leqslant m\}$. F_m^i is a closed subset of X_i, and $(F_0^i, E_1^i, F_1^i, \ldots, E_m^i, F_m^i)$ is an admissible sequence of $G(\mathbf{DC}, X_i)$. s_i is the stationary winning strategy of player one in $G(\mathbf{DC}, X_i)$, so $s_i(F_m^i) \in \mathbf{DC}$. We let $E_{m+1}^i = s_i(F_m^i) = \bigcup \mathcal{F}_{m+1}^i$, where $\mathcal{F}_{m+1}^i$ is a discrete family of compact subsets of X_i. We denote $\mathcal{F}_{m+1}^{i'} = \{F'\colon F' = F \setminus \bigcup\{\phi(d)\colon d \in \bigcup\{D_p\colon p \leqslant m\}\}, F \in \mathcal{F}_{m+1}^i\}$. Let $A_{m+1}^i = \{x\colon \mathcal{F}_{m+1}^{i'}$ is not locally finite at $x\}$. We see that A_{m+1}^i is a closed subset of X, $A_{m+1}^i \subset X \setminus X_i$, and $A^i_{m+1} \cap (\bigcup\{\phi(d)\colon\ d \in \bigcup\{D_p\colon\ p \leqslant m\}\}) = \emptyset$. Thus there exists a closed discrete subset $D^{i*}_{m+1} \subset A^i_{m+1}$, such that $A^i_{m+1} \subset \bigcup\{\phi(d)\colon\ d \in D^{i*}_{m+1}\}$. Let $\mathcal{F}^{i*}_{m+1} = \{F^*\colon\ F \in \mathcal{F}^i_{m+1}\}$, where $F^* = F' \setminus \{\phi(d)\colon\ d \in D^{i*}_{m+1}\}$. So F^* is a compact subset of X for each $F \in \mathcal{F}^i_{m+1}$. $\mathcal{F}^{i*}_{m+1}$ is a locally finite family of X. Thus there is a finite subset $C_F \subset F^*$, such that $F^* \subset \bigcup\{\phi(d)\colon\ d \in C_F\}$.

Let $D^i_{m+1} = (\bigcup\{C_F\colon\ F \in \mathcal{F}^i_{m+1}\}) \cup D^{i*}_{m+1}$. The set $\bigcup\{C_F\colon\ F \in \mathcal{F}^i_{m+1}\}$ is a closed discrete subset of X following the locally finite property of $\mathcal{F}^{i*}_{m+1}$. So we see that D^i_{m+1} is a closed discrete subset of X. Let $D_{m+1} = \bigcup\{D^i_{m+1}\colon\ 1 \leqslant i \leqslant n+1\}$. So D_{m+1} is a closed discrete subset of X, and $D_{m+1} \cap (\bigcup\{\phi(d)\colon\ d \in \bigcup\{D_p\colon\ p \leqslant m\}\}) = \emptyset$. Let $D = \bigcup\{D_m\colon\ m \in N\}$. In the following, we will prove that $X = \bigcup\{\phi(d)\colon\ d \in D\}$.

Suppose there is some $x \in X$, such that $x \notin \bigcup\{\phi(d)\colon\ d \in D\}$. Thus there is some $1 \leqslant i \leqslant n+1$, such that $x \in X_i$. $F^i_m = X_i \setminus \bigcup\{\phi(d)\colon\ d \in \bigcup\{D_p\colon\ p \leqslant m\}\}$. So $x \in F^i_m$. $s_i(F^i_m) \subset F^i_m$, $s_i(F^i_m) \subset \bigcup\{\phi(d)\colon\ d \in D^i_{m+1}\} \subset \bigcup\{\phi(d)\colon\ d \in D_{m+1}\} \subset \bigcup\{\phi(d)\colon\ d \in \bigcup\{D_p\colon\ p \leqslant m+1\}\}$. So $F^i_{m+1} \cap s_i(F^i_m) = \emptyset$. Thus $(F^i_0, E^i_1, F^i_1, \ldots, E^i_m, F^i_m, \ldots)$ is a play of $G(\mathbf{DC}, X_i)$. Thus $\bigcap\{F^i_m\colon\ m \in N\} = \emptyset$. This contradicts with $x \in \bigcap\{F^i_m\colon\ m \in N\}$. So $X = \bigcup\{\phi(d)\colon\ d \in D\}$. Since $D_{m+1} \cap (\bigcup\{\phi(d)\colon\ d \in \bigcup\{D_p\colon\ p \leqslant m\}\}) = \emptyset$. So we know that D is a closed discrete subset of X. Thus X is a D-space. □

A regular space X is a scattered space, if for any non-empty closed subset F of X, there is a point x with a compact neighborhood contained in F (cf. [15]).

Lemma 8. *(See [16].) If X is a regular subparacompact* **C***-scattered space, then X is a* **DC***-like space.*

Corollary 9. *If $X = \bigcup\{X_i\colon\ 1 \leqslant i \leqslant k\}$, and X_i is a regular subparacompact* **C***-scattered space for each $1 \leqslant i \leqslant k$. Then X is a D-space.*

By results of [12], we know that every θ-refinable locally compact space is a **DC**-like space. So we have the following corollary.

Corollary 10. *If $X = \bigcup\{X_i\colon\ 1 \leqslant i \leqslant k\}$, and X_i is a θ-refinable locally compact space for each $1 \leqslant i \leqslant k$. Then X is a D-space.*

Acknowledgement

The author would like to thank the referee for his (or her) valuable suggestions which greatly improved the paper.

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